Int. J. Solids Structures. 1975, Vol. 11, pp. 139–152. Pergamon Press. Printed in Gt. Britain.

BENDING OF THIN CIRCULAR RINGS

SHUN CHENG†
Department of Engineering Mechanics, University of Wisconsin, Madison, Wisconsin 53706, U.S.A.

and

N. J. HOFF‡
Department of Aeronautics and Astronautics, Stanford University, Stanford, California 94305, U.S.A.

(*Received* 17 *December* 1973; *revised* 10 *May* 1974)

Abstract—The basic equations for the bending of circular rings are deduced from a set of accurate equations for circular cylindrical shells. The advantages in using these differential equations as compared with the customary energy method are shown through examples. It turns out that solutions of these equations can be as easily obtained as solutions of the well-known differential equation for straight beams. It is also shown that the center line of the ring is essentially inextensible, which is assumed *ab initio* in the classical ring theory.

NOMENCLATURE

a radius of midsurface
b width of the ring
c^2 $h^2/12a^2$
h wall thickness
I moment of inertia
P load
u, v, w axial, circumferential, and radial displacements of midsurface (Fig. 3)
x, y, z axial, circumferential, and radial coordinates
X, Y, Z surface loads per unit area in axial, tangential and radial directions (Fig. 2)
$D = \frac{Eh^3}{12(1-\nu^2)}$ flexural rigidity [8, 9]
$K = Eh/1-\nu^2$ = extensional rigidity [8, 9]
E, G, ν Young's modulus, shear modulus and Poisson's ratio
M_1, M_2, M_{12}, M_{21}, N_1, N_2, S_1, S_2 stress couples and resultants per unit length (Fig. 2, 3) [8, 9]
Q_1, Q_2 transverse stress resultants per unit length (Fig. 2) [8, 9]
$M = bM_2$ = bending moment
α, θ dimensionless midsurface coordinates along the lines of curvatures (Fig. 1, 2)
$\alpha = x/a$, $\theta = y/a$
$\epsilon_1, \epsilon_2, \epsilon_{12}$ normal and shearing strains in midsurface [8, 9]
η_1, η_2, τ bending and twisting strains [8, 9]
$\sigma_\alpha, \sigma_\beta, \tau_{\alpha\beta}$, $e_\alpha, e_\beta, e_{\alpha\beta}$ stresses and strains of an arbitrary point [9]
$\omega_\alpha, \omega_\theta$ components of rotation w.r.t. parametric lines α and θ respectively [9]
$\omega = \omega_\alpha$
$\nabla^2 = (\partial^2/\partial\alpha^2) + (\partial^2/\partial\theta^2)$.

INTRODUCTION

The problem of the bending of a thin ring in its plane was solved nearly a century ago[1, 2] and because of its importance in practical applications, has been a major subject in many texts on structural analysis[3] and in other mechanics courses[4, 5]. In solving the problem, it may be noted that two features are unique—the use of Castigliano's theorem and the adoption of the

†Professor, Dept. of Engineering Mechanics.
‡Professor, Dept. of Aeronautics and Astronautics.

notion of the inextensibility of the center line of the ring [4–7]. Although this notion has been commonly accepted, its general validity has not yet been established except as illustrated through some specific examples [7]. As for the method traditionally used in solving the ring problem, it is a striking contrast to problems of bending of straight beams for which, as we know, differential equation rather than the energy method is often used in the texts [7]. Surprisingly, for ring problems, solutions by differential equations have not received the attention they truly deserve. In fact, it turns out that the differential equation method is not only more direct but also easier to grasp and understand then the energy method based on Castigliano's theorem. Solutions of these equations can be as readily obtained as solutions of straight beam problems. The lack of such an approach is, perhaps due to the fact that the three basic equations required for solving the ring problem are scattered under different headings in the literature and have not been tied together for ready application to bending problems. These equations are presented in this paper. They can be deduced in a fairly straightforward manner from the basic equations of an accurate shell theory [8, 9, 10]. The validity of the notion of inextensibility of the center line of the ring will also be established. The advantages in using these equations will be shown through several illustrative examples.

ANALYSIS

As is well known the classical shell theory is based on the Kirchhoff assumption: material normals to the undeformed middle surface remain normal to the deformed middle surface of the shell and retain their length. The formulation of the basic equations for thin elastic shells and, in particular, circular cylindrical shells, due to their importance in application and their exhibition of nearly every type of behavior found in general shell theory, has received repeated attention in the literature [8, 9, 11, 12]. As for the question of the validity of the Kirchhoff assumption, John recently has given a mathematical proof of what was heretofore only intuitive assumption of the classical shell theory [14]. John's results were reexamined by Koiter [15]. In the case of the bending of thin rings, which is a special case of circular cylindrical shells, the Kirchhoff assumption may be phrased as "plane sections remain plane after deformation."

In a recent paper [8] a set of accurate equations which govern the deformation of circular cylindrical shells is presented. It is shown [8, 9] that these equations are as accurate as they can be within the scope of the Kirchhoff assumption. In order to make the present paper reasonably self-contained, these equations derived in [8], which for the reasons just mentioned will be employed to deduce the basic equations for the bending of circular rings, are again presented in the Appendix. (It is shown in [8] that all the known equations, such as Donnell, Novozhilov, Morley, extended Donnell and others can be readily obtained from the fourth order equation derived in [8]. Also its solutions can be easily expressed in simple closed forms [8, 9]).

Aside from John's rigorous mathematical proof, the validity of the assumption "plane sections remain plane" for the bending of curved bars may also be visualized in a less rigorous and perhaps more physical way. We recall the three simple but fundamentally important examples given in Timoshenko and Goodier's *Theory of Elasticity* [13]. These examples all belong to plane stress problems in elasticity theory.
They are:

(1) Pure bending of curved bars (p. 72 [13]).

(2) A curved bar bent by a tangential force applied at one end in radial direction with the other end of the bar constrained (p. 83 [13]).

(3) A curved bar bent by a normal force applied at one end with the other end constrained (p. 88 [13]).

Having the solutions of these three problems, the solutions for general loading can be obtained by superposition. Comparison of results [13] from the exact elasticity theory of plane stress with those from the simple theory, based on the hypothesis that plane sections remain plane during bending, shows that for these three examples the simple theory does give very satisfactory results even when the ratio of the width of the bar to its radius is not very small. These results further justify the basic Kirchhoff hypothesis for the bending of thin shells and curved bars.

DERIVATION OF BASIC EQUATIONS

When a ring is loaded by forces applied at the boundary, parallel to the plane of the ring, the stress components are zero on both faces of the ring (Fig. 1). Such a state of stress is called plane stress. Thus according to plane stress theory, N_1, M_1, S_1, S_2, M_{12}, M_{21} and Q_1 in equations (25) may be set equal to zero (same notations as in [8–10]). Setting $N_1 = 0$ and $M_1 = 0$, we obtain

$$-\nu w + c^2\frac{\partial^2 w}{\partial\alpha^2} = \frac{\partial^2 w}{\partial\alpha^2} + \nu\frac{\partial^2 w}{\partial\theta^2} = \frac{\partial u}{\partial\alpha} + \nu\frac{\partial v}{\partial\theta}$$

or

$$\frac{\partial^2 w}{\partial\alpha^2} = \frac{-\nu}{1-c^2}\left(\frac{\partial^2 w}{\partial\theta^2} + w\right). \tag{1}$$

From $S_1 = 0$ and $S_2 = 0$

$$\frac{\partial v}{\partial\alpha} = \frac{\partial^2 w}{\partial\alpha\partial\theta} = -\frac{\partial u}{\partial\theta} \tag{2}$$

is obtained.

Setting M_{12}, M_{21} and Q_1 equal to zero in equation (25) does not yield any equation in addition to equations (1) and (2) already obtained. Applying equations (1) and (2), the rest of the equations in equation (25) may be written as

$$\begin{aligned} N_2 &= \frac{K}{a}\left[(1-\nu^2)\left(\frac{\partial v}{\partial\theta} + w\right) + c^2\left(1 - \frac{\nu^2}{1-c^2}\right)\left(\frac{\partial^2 w}{\partial\theta^2} + w\right)\right] \\ M_2 &= \frac{D}{a^2}\left(1 - \frac{\nu^2}{1-c^2}\right)\left(\frac{\partial^2 w}{\partial\theta^2} + w\right) \\ Q_2 &= \frac{-D}{a^3}\left(1 - \frac{\nu^2}{1-c^2}\right)\left(\frac{\partial^3 w}{\partial\theta^3} + \frac{\partial w}{\partial\theta}\right) = \frac{-1}{a}\frac{\partial M_2}{\partial\theta}. \end{aligned} \tag{3}$$

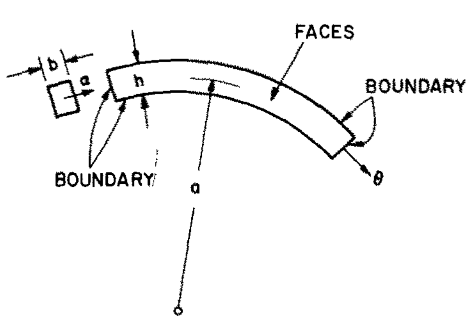

Fig. 1. Circular ring.

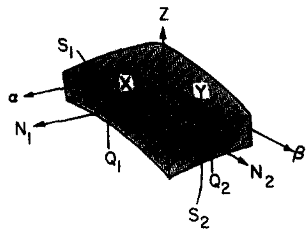

Fig. 2. Stress resultants and surface loads acting on differential element.

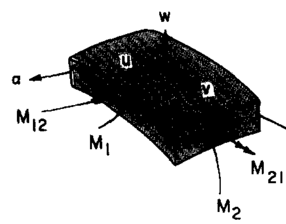

Fig. 3. Stress couples acting on differential element and midsurface displacements.

It is now appropriate to note that in the theory of thin rings $h/a \leq 1/10$, thus $c^2 < 10^{-3}$ and c^2 is always a small number as compared with unity. Thus, equation (3) may be written as

$$N = N_2 = \frac{Eh}{a}\left[\frac{\partial v}{\partial \theta} + w + c^2\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)\right] \tag{4}$$

$$M_2 = Ehc^2\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) \tag{5}$$

$$Q = Q_2 = \frac{-Eh}{a}c^2\left(\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta}\right) = \frac{-1}{a}\frac{\partial M_2}{\partial \theta}. \tag{6}$$

Three of the five equations of equilibrium (26) are identically satisfied and the other two equations of equilibrium are

$$\begin{gathered} N - \frac{\partial Q}{\partial \theta} - aZ = 0 \\ \frac{\partial N}{\partial \theta} + Q + aY = 0. \end{gathered} \tag{7}$$

Substituting N and Q from equations (4) and (6) into equations (7) yields

$$\frac{\partial v}{\partial \theta} = -w - c^2\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 w + \frac{a^2}{Eh}Z \tag{8}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta} + w\right) = -\frac{a^2}{Eh}Y. \tag{9}$$

As the derivatives of the normal deflection w in case of small deformations are of the same order of magnitude as the w itself, which can also be verified directly from the solution of a problem, and the coefficient c^2 is always small as compared with unity, the second term on the right side of equation (8) may be neglected as compared with the first term on the same side of the equation without any significant loss in accuracy of the final solutions. Then equation (8) may be written in the form

$$\frac{\partial v}{\partial \theta} + w = \frac{a^2}{Eh} Z. \tag{10}$$

In the absence of surface load Y, which is the usual case, equation (9) becomes

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta} + w\right) = 0. \tag{11}$$

CONDITIONS FOR THE INEXTENSIBLE DEFORMATION OF CIRCULAR RINGS

In the literature, as mentioned previously, the ring theory is based on the assumption that the center line of a ring is inextensible and the strain energy is primarily due to bending. From equations (10) and (11) the validity of this assumption can now be established. When a ring is under external loads applied at the ends of the ring or under concentrated surface loads or both, Z is zero everywhere except where concentrated surface loads are applied and equation (10) reduces to

$$\frac{\partial v}{\partial \theta} + w = 0. \tag{12}$$

This result shows that the tangential strain of the center line of a ring is zero ($\epsilon_2 = 0$) as seen from equation (24) and thus establishes the validity of the basic assumption of the classical ring theory.

As will be illustrated later in the examples, the radial displacement of the ring can be obtained from equation (5), and equation (12) is used primarily for determining the tangential displacement v. Equation (5) shows that the order of magnitude of the radial displacement w is $(1/c^2)(M_2/Eh)$. If the order of magnitude of the term Za^2/Eh in equation (10) is taken as 0(1), then the term $(1/c^2)(M_2/Eh)(M_2 \sim Za^2$, see examples) is of $0(1/c^2)$. Thus the term Za^2/Eh in equation (10) has a magnitude of $0(c^2)$ relative to the w term in the same equation. When the tangential displacement v, provided $v \neq 0$, is calculated from equation (10), the term Za^2/Eh makes a negligible contribution to the tangential displacement v and thus may be neglected. Hence in the case $Z \neq 0$, the condition for inextensible deformation (12) is again obtained.

BASIC EQUATIONS FOR THE BENDING OF CIRCULAR RINGS

The three basic equations for the bending of circular rings are as follows:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + w = \frac{Ma^2}{EI} \tag{13}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} + w = 0 \tag{14}$$

$$\omega = \frac{1}{a}\left(\frac{\mathrm{d}w}{\mathrm{d}\theta} - v\right). \tag{15}$$

Where $I = bh^3/12$ = moment of inertia, b = width of the ring and M is the bending moment ($M = bM_2$). M, w and v are now assumed to be functions of θ only which is the case for bending of thin rings. Equation (13) is obtained from equation (5). Equation (14) represents the condition that the center line of the ring is inextensible as established in the last section ($\epsilon_2 = 0$). Equation (15) represents the rotation of radial cross sections of the ring about its longitudinal axis (α coordinate line) as given by equation (21). It will be shown in the next section that these three equations (13), (14) and (15) are sufficient for the complete solution of problems of the bending of circular rings. The tangential force N and the shearing force Q may be readily determined from equations (4) and (6) or from the conditions of static equilibrium of the ring. The normal stress σ_θ of the ring may be obtained from equations (22), (23), (4), (1) and (13) as

$$\sigma_\theta = \frac{N}{h} - \frac{Ma}{I}\left[\frac{z}{a+z}\left(1 - \frac{\nu^2 z}{(1-\nu^2)a}\right) + \frac{c^2}{1-\nu^2}\right]. \tag{16}$$

When the thickness of the ring is small in comparison with a, we obtain

$$\sigma_\theta = \frac{Nb}{A} - \frac{Mz}{I}\frac{a}{a+z} \tag{17}$$

where $A = bh$ = cross sectional area of the ring.

It may be noted that for an infinitely large radius a the preceding equations (13), (14), (15) and (17) reduce, as they should, to the well-known equations for straight beam

$$\frac{d^2w}{dx^2} = \frac{M}{EI}$$

$$\omega = \frac{dw}{dx}, \quad \sigma_\theta = \frac{Nb}{A} - \frac{Mz}{I}.$$

We have mentioned in the introduction that a unique feature of the solution of ring problems in the literature is the use of the energy method in marked contrast to problems of straight beams, for which differential equations are often used. In the next section we shall show that solutions for bending problems of circular rings can be readily obtained through the use of equations (13), (14) and (15). These equations are particularly useful when the deflection curve of the ring is required. The integrations of equations (13), (14) and (15) are as simple as that of the equation for the deflection of straight beams.

For many practical applications the general solution of equation (13) can be written in the form

$$w = A\cos\theta + B\sin\theta + C_1\theta\cos\theta + C_2\theta\sin\theta + w_0. \tag{18}$$

Substituting the preceding expression (16) into equation (14), the tangential displacement v may be expressed as

$$v = -A\sin\theta + B\cos\theta - C_1(\cos\theta + \theta\sin\theta) - C_2(\sin\theta - \theta\cos\theta) - w_0\theta + v_0 \tag{19}$$

where constants A, B, C_1, C_2, w_0 and v_0 can be determined from equation (13) and the end conditions of the ring as will be illustrated in the following section.

Example 1

Consider one quadrant of the ring built in at the lower end and loaded at the upper end by a vertical load P (Fig. 4). The bending moment at any cross section is

$$M = -Pa \cos \theta.$$

Taking $w = A \cos \theta + B \sin \theta + C_1 \theta \cos \theta + C_2\theta \sin \theta + A_0$ and substituting in (13), we obtain

$$C_1 = A_0 = 0, \text{ and } C_2 = \frac{-Pa^3}{2EI}.$$

Thus

$$w = A \cos \theta + B \sin \theta + C_2 \theta \sin \theta.$$

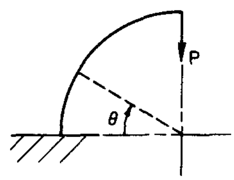

Fig. 4. Quadrant of a circular ring with radial load.

Constants A and B can be determined from the following boundary conditions of the ring.

$$w = 0, \quad \frac{dw}{d\theta} = 0 \text{ for } \theta = 0.$$

From these two conditions we obtain

$$A = B = 0,$$

$$w = \frac{-Pa^3}{EI} \theta \sin \theta.$$

The tangential displacement v is found from

$$\frac{dv}{d\theta} + w = 0$$

or

$$\frac{dv}{d\theta} = \frac{Pa^3}{EI} \theta \sin \theta.$$

Thus

$$v = \frac{Pa^3}{EI}(\sin \theta - \theta \cos \theta) + v_0.$$

Applying the end condition $v = 0$ for $\theta = 0$ we obtain $v_0 = 0$ and

$$v = \frac{Pa^3}{EI}(\sin\theta\cos\theta).$$

Example 2

Consider one quadrant of the ring built in at the upper end and loaded at the other end by a vertical load P (Fig. 5). The bending moment at any cross section is

$$M = -Pa(1 - \cos\theta).$$

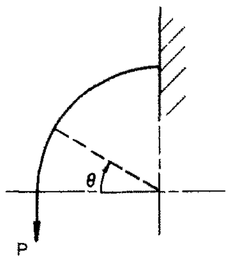

Fig. 5. Quadrant of a circular ring with tangential load.

Taking $w = A\cos\theta + B\sin\theta + C_1\theta\cos\theta + C_2\theta\sin\theta + A_0$ and substituting in (13) we obtain

$$C_1 = 0, \quad C_2 = \frac{Pa^3}{2EI}, \quad A_0 = \frac{-Pa^3}{EI}.$$

Applying the following end conditions

$$\frac{dw}{d\theta} = 0, \text{ for } \theta = \frac{\pi}{2}$$

and

$$w = 0, \text{ for } \theta = \frac{\pi}{2}$$

we obtain

$$C_2 = A \text{ and } B = -\frac{\pi}{2}C_2.$$

Thus

$$w = \frac{Pa^3}{2EI}\left[\cos\theta + \left(2 - \frac{\pi}{2}\right)\sin\theta + \theta\sin\theta - 2\right].$$

From $dv/d\theta = -w$ there follows

$$v = \frac{Pa^3}{2EI}\left[2\theta + \theta\cos\theta - \left(\frac{\pi}{2} - 2\right)\cos\theta - 2\sin\theta - 2 + \frac{\pi}{2}\right] + v_0.$$

Since $v = 0$ for $\theta = \pi/2$, we have

$$v_0 = (4 - 3\pi)\frac{Pa^3}{4EI},$$

and

$$v = \frac{Pa^3}{2EI}\left(2 - \pi + 2\theta + \theta\cos\theta - \frac{\pi}{2}\cos\theta - 2\sin\theta + 2\cos\theta\right).$$

From equation (15) there follows

$$\omega = \frac{1}{a}\frac{dw}{d\theta} - \frac{v}{a} = \frac{Pa^2}{2EI}(\pi - 2 - 2\theta + 2\sin\theta).$$

Example 3

Consider a semicircular ring built in at one end and loaded at the other end (Fig. 6). The bending moment at any cross section is

$$M = Pa(1 - \cos\theta).$$

Taking $w = A\cos\theta + B\sin\theta + C\theta\sin\theta + w_0$ and substituting w into equation (13) yields

$$C = -\frac{w_0}{2} \text{ and } w_0 = \frac{Pa^3}{EI}.$$

End conditions are $w = 0$ and $dw/d\theta = 0$ at $\theta = \pi/2$.
Hence, we find that

$$A = w_0, \quad B = \frac{\pi}{2}w_0$$

and

$$w = \frac{Pa^3}{EI}\left(1 + \cos\theta + \frac{\pi}{2}\sin\theta - \frac{1}{2}\theta\sin\theta\right).$$

From $dv/d\theta + w = 0$, yields

$$v = -w_0\left(\theta + \frac{1}{2}\sin\theta - \frac{\pi}{2}\cos\theta + \frac{1}{2}\theta\cos\theta\right) + v_0.$$

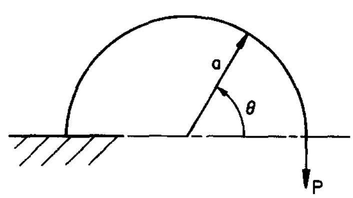

Fig. 6. Semicircular wing with one section clamped.

Since $v = 0$ for $\theta = \pi/2$, we have $v_0 = \pi Q$ and

$$v = \frac{Pa^3}{2EI}[2\pi - 2\theta - \sin\theta + (\pi - \theta)\cos\theta].$$

Example 4

Consider a semicircular ring clamped at both ends and loaded along the axis of symmetry with a concentrated load P (Fig. 7). The bending moment at an arbitrary cross section is

$$M = \frac{P}{2}a(1 - \cos\theta) - Ha\sin\theta.$$

Taking

$$w = A\cos\theta + B\sin\theta + C_1\theta\cos\theta + C_2\theta\sin\theta + w_0$$

and substituting w in (13) yields

$$w_0 = \frac{Pa^3}{2EI},\quad C_1 = \frac{a^3H}{2EI} \text{ and } C_2 = \frac{-Pa^3}{4EI}.$$

From the end condition

$$w = 0 \text{ for } \theta = 0, \text{ there follows } A = -w_0.$$

From equation (14) we obtain

$$v = -[A\sin\theta - B\cos\theta + C_1(\cos\theta + \theta\sin\theta) + C_2(\sin\theta - \theta\cos\theta) + w_0\theta] + v_0.$$

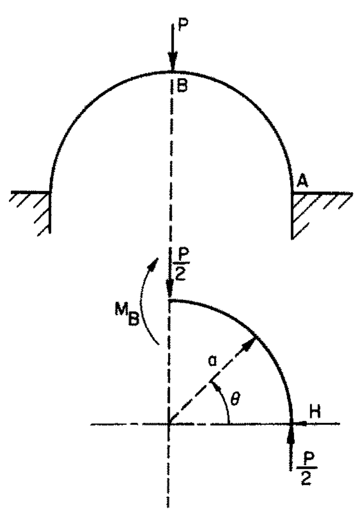

Fig. 7. Semicircular ring with two sections clamped.

From the end conditions

$$v = 0 \text{ for } \theta = 0 \text{ and } \theta = \frac{\pi}{2}$$

we obtain

$$B - C_1 + v_0 = 0 \tag{a}$$

and

$$A + \frac{\pi}{2} C_1 + C_2 + \frac{\pi}{2} w_0 - v_0 = 0. \tag{b}$$

Applying the end condition

$$\omega = 0, \text{ for } \theta = \frac{\pi}{2}$$

we obtain from equation (15)

$$2C_2 + \frac{\pi}{2} w_0 - v_0 = 0. \tag{c}$$

Solution of the preceding three algebraic equations yields

$$H = \frac{P}{\pi}$$

$$v_0 = 2C_2 + \frac{\pi}{2} w_0 = \frac{Pa^3}{2EI}\left(\frac{\pi}{2} - 1\right)$$

$$B = C_1 - v_0 = \frac{Pa^3}{2EI}\left(\frac{1}{\pi} - \frac{\pi}{2} + 1\right).$$

Thus the radial displacement at point B is

$$w_B = B + \frac{\pi}{2} C_2 + w_0 = \frac{(4 + 8\pi - 3\pi^2)}{8\pi} \frac{Pa^3}{EI}.$$

BASIC EQUATIONS FOR LONG TUBES

As a secondary result of our study we derive the governing equations for a long circular tube under the action of lateral loads uniformly distributed along the axis of the cylinder. In this case it is a state of plane strain. Thus the displacement along the axis of the tube is zero and displacements v and w are functions of θ only. Following procedures similar to the deductions of the basic equations for the ring, the following equations for the bending of long tubes can be obtained

$$\frac{d^2 w}{d\theta^2} + w = \frac{12(1 - \nu^2) a^2 M}{Eh^3}$$

$$\frac{dv}{d\theta} + w = 0$$

$$\omega = \frac{1}{a}\left(\frac{dw}{d\theta} - v\right)$$

where M represents bending moment per unit length along the axis of the tube.

CONCLUSIONS

Although the ring problem was solved nearly a century ago, its solution by differential equations has not received the attention it truly deserves. Also the general validity of the basic assumption in ring theory, namely that the center line of the ring is inextensible, has not been established except as illustrated through some specific examples[7]. Because of the importance of the ring problem in practical applications and the advantages of the differential equations over the customary Castigliano's theorem, we presented the three basic differential equations for the complete solution of the ring problem. Further, in the paper the validity of the notion of the inextensibility of the center line of the ring is established. As a secondary result of our study the governing equations for long circular tubes are also presented.

REFERENCES

1. B. Rayleigh, *The Theory of Sound.* Dover, New York (1945).
2. H. Lamb, On the Flexure and the Vibrations of a Curved Bar. *Proc. London Math. Soc.* **19**, 365 (1888).
3. N. J. Hoff, *The Analysis of Structures.* Wiley, New York (1956).
4. C. B. Biezeno and R. Grammel, *Engineering Dynamics*, Vol. II. Blackie and Son, London (1956).
5. F. B. Seely and J. O. Smith, *Advanced Mechanics of Materials*, 2nd ed. Wiley, New York (1952).
6. J. A. Van den Broek, *The Elastic Energy Theory.* Wiley, New York (1948).
7. S. P. Timoshenko, *Strength of Materials*, Part I, 3rd ed. Van Nostrand, Cincinnati, Ohio (1955).
8. Shun Cheng, Accurate Fourth Order Equation for Circular Cylindrical Shell *J. Engng. Mech. Div., Proc. ASCE*, **98** (No. EM3), 641–656 (June 1972).
9. Shun Cheng, On an Accurate Theory for Circular Cylindrical Shells, *J. Appl. Mech. Trans. ASME*, **40**, **E** (No. 2), 582–588 (June 1973).
10. W. Flügge, *Stress in Shells.* Springer, Berlin (1962).
11. M. W. Johnson and E. Reissner, On the Foundations of the Theory of Thin Elastic Shells, *J. Math. Phys.* **37**, 371–392 (1959).
12. P. M. Naghdi, On the Theory of Thin Elastic Shells, *Q. appl. Math.* **14** (1957).
13. S. P. Timoshenko, and J. N. Goodier, *Theory of Elasticity*, 3rd ed. McGraw-Hill, New York (1970).
14. F. John. Refined Interior Equations for the Thin Elastic Shells, *Communications on Pure and Applied Mathematics* **XXIV**, 583–615 (1971).
15. W. T. Koiter, A Comparison Between John's Refined Interior Shell Equations and Classical Shell Theory, with an Appendix by F. John, *J. appl. Math. Phys. (ZAMP)* **20**, 642–652 (1969).

APPENDIX

The three displacement components u_α, u_β, and u_z of an arbitrary point of the shell can be written as[9, 10]

$$u_\alpha = u + z\omega_\theta, \quad u_\theta = v - z\omega_\alpha, \quad u_z = w \tag{20}$$

where the components of rotation ω_α, ω_θ about the parmetric lines α, θ are [9, 10]

$$\omega_\alpha = \frac{1}{a}\frac{\partial w}{\partial \theta} - \frac{v}{a}, \quad \omega_\theta = \frac{-1}{a}\frac{\partial w}{\partial \alpha} \tag{21}$$

and the stress–strain relations can be taken in the form

$$\sigma_\alpha = \frac{E}{1-\nu^2}(e_\alpha + \nu e_\theta), \sigma_\theta = \frac{E}{1-\nu^2}(e_\theta + \nu e_\alpha), \tau_{\alpha\theta} = Ge_{\alpha\theta}. \tag{22}$$

The components of strain e_α, e_β, and $e_{\alpha\beta}$ at an arbitrary point of the shell are related to the midsurface displacements by [9, 10]

$$e_\alpha = \frac{1}{a}\left(\frac{\partial u}{\partial \alpha} - \frac{z}{a}\frac{\partial^2 w}{\partial \alpha^2}\right)$$

$$e_\theta = \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + w\right) - \frac{z}{a(a+z)}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right) \tag{23}$$

$$e_{\alpha\theta} = \frac{1}{a+z}\left[\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \alpha} + 2\frac{z}{a}\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \theta}\right) + \left(\frac{z}{a}\right)^2\left(\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \theta}\right)\right].$$

The strain displacement relations of the middle surface of the shell are [8, 10]:

$$\epsilon_1 = \frac{1}{a}\frac{\partial u}{\partial \alpha}$$

$$\epsilon_2 = \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + w\right)$$

$$\epsilon_{12} = \frac{1}{a}\left(\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \alpha}\right)$$

$$\eta_1 = \frac{-1}{a^2}\frac{\partial^2 w}{\partial \alpha^2}$$

$$\eta_2 = \frac{-1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)$$

$$\tau = \frac{-1}{2a^2}\left(\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \theta}\right). \tag{24}$$

The stress resultants and couples are related to the midsurface displacements through the stress–strain relations as

$$N_1 = \frac{K}{a}\left[\frac{\partial u}{\partial \alpha} + \nu\left(\frac{\partial v}{\partial \theta} + w\right) - \frac{h^2}{12a^2}\frac{\partial^2 w}{\partial \alpha^2}\right]$$

$$N_2 = \frac{K}{a}\left[\frac{\partial v}{\partial \theta} + \nu\frac{\partial u}{\partial \alpha} + w + \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial \theta^2} + w\right)\right]$$

$$S_1 = \frac{K(1-\nu)}{2a}\left[\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \alpha} - \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial \alpha \partial \theta} - \frac{\partial v}{\partial \alpha}\right)\right]$$

$$S_2 = \frac{K(1-\nu)}{2a}\left[\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \alpha} + \frac{h^2}{12a^2}\left(\frac{\partial^2 w}{\partial \alpha \partial \theta} + \frac{\partial u}{\partial \theta}\right)\right]$$

$$M_1 = -\frac{D}{a^2}\left[\frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \theta} - \left(\frac{\partial^2}{\partial \alpha^2} + \nu\frac{\partial^2}{\partial \theta^2}\right) w\right]$$

$$M_2 = \frac{D}{a^2}\left[\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial \alpha^2} + w\right]$$

$$M_{12} = \frac{D(1-\nu)}{a^2}\left[\frac{\partial v}{\partial \alpha} - \frac{\partial^2 w}{\partial \alpha \partial \theta}\right]$$

$$M_{21} = -\frac{D(1-\nu)}{2a^2}\left[\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial \alpha} + 2\frac{\partial^2 w}{\partial \alpha \partial \theta}\right]$$

$$Q_1 = \frac{D}{a^3}\left[\frac{\partial^2 u}{\partial \alpha^2} - \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial \alpha \partial \theta} - \frac{\partial}{\partial \alpha}\nabla^2 w\right] \tag{25}$$

$$Q_2 = \frac{D}{a^3}\left[(1-\nu)\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial w}{\partial \theta} - \frac{\partial}{\partial \theta}\nabla^2 w\right].$$

The coefficient δ [8] which is always small ($\delta \doteq (9/5)\, c^2$) as compared with unity has been omitted in equation (20). The following equations of static equilibrium are well known:

$$\frac{\partial N_1}{\partial \alpha} + \frac{\partial S_2}{\partial \theta} + aX = 0$$

$$\frac{\partial N_2}{\partial \theta} + \frac{\partial S_1}{\partial \alpha} + Q_2 + aY = 0$$

$$-N_2 + \frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \theta} + aZ = 0 \tag{26}$$

$$\frac{\partial M_{12}}{\partial \alpha} - \frac{\partial M_2}{\partial \theta} - aQ_2 = 0$$

$$\frac{\partial M_{21}}{\partial \theta} - \frac{\partial M_1}{\partial \alpha} - aQ_1 = 0.$$